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EXPONENTIAL STABILITY OF THE LINEAR KDV-BBM EQUATION

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Abstract. In this paper, we consider the following linear Korteweg–de Vries–Benjamin Bona Mahony (KdV-BBM) equation on a finite interval.

$$\begin{cases} u_t - a^2 u_{xxt} + u_x + u_{xxx} = 0, & (x,t) \in (0,L) \times (0,\infty), \\ u(0,t) = u(L,t) = u_x(L,t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in (0,L). \end{cases}$$

We show the well-posedness by the semigroup theory. A set of critical length $\mathcal{L}$ is obtained, for which the system possesses conservative solutions. Then we prove the exponentially stability of the associated semigroup when $L \notin \mathcal{L}$ by the frequency domain method.

1. Introduction.

The Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

is a well known nonlinear dispersive partial differential equation which models the propagation of small amplitude long water waves in a uniform channel, as well as some other physical phenomenons. It has been investigated extensively. We refer to papers [16, 17] for a comprehensive review. More recent works can be found, just quote a few, in [5, 6, 7, 15, 19, 20, 21, 22, 23, 24, 26] and the references therein.

The Benjamin-Bona-Mahony equation (BBM)

$$u_t + u_x + uu_x - u_{txx} = 0 \tag{2}$$

was proposed in [3] as an improved model to the KdV equation. Zhang and Zuazua [27] studied a more general version of this equation on finite interval including a space-dependent potential with Dirichlet boundary conditions. They obtained the asymptotic expression of the eigenvalues, and the Riesz basis property for the associated eigenvectors, which implies the exponential stability for the above linearized BBM equation.

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Later, the following general framework was introduced in [9] (also see [4] for a derivation of this equation)

$$u_t - u_{txx} - C_1 u_{xxx} + C_2 u_x + u u_x = 0 \tag{3}$$

where $C_1, C_2 \in \mathbb{R}$.

When $C_1 = -1, C_2 = 1$, equation (3) comes to be the so-called KdV-BBM equation:

$$u_t - u_{txx} + u_{xxx} + u_x + u u_x = 0. \tag{4}$$

For the KdV-BBM equation, Li and Liu [13] established the well-posedness over finite interval. Asokan and Vinodh [1] obtained its exact solution over the real line by tanh-coth method. Numerical simulation was performed in [8].

In this paper, we consider the linear KdV-BBM equation over a finite interval $[0, L]$ with point dissipation:

$$\begin{cases} u_t - a^2 u_{txx} + u_x + u_{xxx} = 0, x \in (0, L), u(x, 0) = \phi_0(x), x \in [0, L], t \geq 0, \\ u(0, t) = 0, u(L, t) = 0, u_x(L, t) = 0. \end{cases} \tag{5}$$

Our main interest is the stability of (5). For the corresponding linear KdV equation (when $a = 0$), Perla Menzala and Vasconcellos [15] proved exponential stability based an observability inequality [19], except for a set of critical length L which leads to conservative solution of the system. We should also mention the work of Russell and Zhang [20, 21] on the smoothing and stabilizability of the related third order dispersion equation. It is unknown whether the additional term $-a^2 u_{txx}$ to the linear KdV equation can retain the exponential stability since its energy space changes from L^2 to H^1.

This paper is organized as follows. Section 2 deals with the associated semigroup generation. A set of critical length L and its characterization are discussed in Section 3. The exponential stability of (5) when $L \notin \mathcal{L}$ will be proved in Section 4.

2. **Wellposedness.** Let

$$\mathcal{H} = H_0^1(0, L) \tag{6}$$

with the inner product

$$\langle u,\ v\rangle_{\mathcal{H}} = \langle u,\ v\rangle + a^2 \langle u',\ v'\rangle. \tag{7}$$

Hereafter, $\langle \cdot,\ \cdot\rangle$ and $\|\cdot\|$ represent the complex inner product and the norm in $L^2(0, L)$. We also denote $\mathcal{H}^{-1} = (H_0^1)'$, the dual space of $\mathcal{H}$.

Define an operator

$$A_0 u = u - a^2 u'', \qquad \mathcal{D}(A_0) = H^2(0, L) \cap H_0^1(0, L), \tag{8}$$

which is self-adjoint and positive definite in $L^2(0, L)$. Moreover,

$$\|u\|_{\mathcal{H}} = \|A_0^{\frac{1}{2}} u\|, \qquad \|v\|_{\mathcal{H}^{-1}} = \|A_0^{-\frac{1}{2}} v\|. \tag{9}$$

Hence, A_0 is also an isometric isomorphism from $\mathcal{H}$ to $\mathcal{H}^{-1}$.

We now define the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ by

$$Au = A_0^{-1} T u \tag{10}$$

with

$$\mathcal{D}(A) = \mathcal{D}(T) = \left\{ u \in H^2(0, L) \cap H_0^1(0, L) \;\middle|\; u'(L) = 0 \right\} \tag{11}$$

where $Tu = -u''' - u'$.

Thus, the linear KdV-BBM equation (5) can be rewritten as the following first order evolution equation on the Hilbert space $\mathcal{H}$

$$\begin{cases} \frac{du}{dt} = Au, \\ u|_{t=0} = u_0(x). \end{cases} \tag{12}$$

Theorem 2.1. *Assume that $L \neq 2n\pi$ for all $n \in Z^+$. Then the operator A generates a C_0 semigroup of contractions on $\mathcal{H}$. Furthermore, A^{-1} is compact.*

Proof. For $u \in \mathcal{D}(A)$, we have

$$\begin{aligned} Re\langle Au,\ u\rangle_{\mathcal{H}} &= Re\langle Tu,\ u\rangle \\ &= Re\langle u'' + u,\ u'\rangle \\ &= -\frac{1}{2}|u'(0)|^2 \leq 0. \end{aligned} \tag{13}$$

Hence, A is dissipative. Next, we consider the equation

$$Au = f, \qquad f \in H_0^2(0, L)$$

which is equivalent to

$$\begin{cases} -u'''(x) - u'(x) = f(x) - a^2 f''(x), \\ u(0) = u(L) = u'(L) = 0. \end{cases} \tag{14}$$

By a straight forward calculation (it can also be seen from the solution to equation (39) when $s = 0$ in the proof of Theorem 4.3), we obtain

$$u(x) = \int_L^x [1-(1+a^2)\cos(x-\xi)]f(\xi)d\xi + \frac{1-\cos(x-L)}{1-\cos L}\int_0^L [1-(1+a^2)\cos\xi]f(\xi)d\xi. \tag{15}$$

Since H_0^2 is dense in $\mathcal{H}$, the above also holds for $f \in \mathcal{H}$. Moreover, it is easy to check

$$\|u\|_{\mathcal{H}} \leq M\|f\| \leq M'\|f\|_{\mathcal{H}} \tag{16}$$

for some constants $M, M' > 0$. Thus, $A^{-1} \in \mathcal{L}(\mathcal{H})$. This proved that $0 \in \rho(A)$. By the Theorem 1.2.4 in [14], we conclude that A is the infinitesimal generator of an C_0 semigroup of contractions on $\mathcal{H}$. Since the embedding $\mathcal{H} \hookrightarrow L^2(0, L)$ is compact, there is a convergent subsequence in $L^2(0, L)$ for any f_n bounded in $\mathcal{H}$. By the first inequality in (16), $A^{-1}f_n$ has a convergent subsequence in $\mathcal{H}$. Therefore, A^{-1} is compact. □

Remark 2.2. The condition $L \neq 2n\pi$ is not a necessary condition for A to be the infinitesimal generator of a C_0 semigroup of contractions on $\mathcal{H}$. It is known that 0 is an eigenvalue of A if $L = 2n\pi$. Since our main interest is the exponential stability of the system, this condition is unavoidable. On the other hand, it does simplify the proof of the wellposedness.

3. **Set of critical length.**

In this section, similar to the study of linear KdV equation, we will identify a set of the critical Length L for the linear KdV-BBM equation, which is related to the existence of eigenvalues of A on the imaginary axis.

Theorem 3.1. *There exists $\lambda \in \mathbb{C}$, and $0 \neq u \in H^3(0, L)$ satisfying*

$$\begin{cases} \lambda u - \lambda a^2 u'' + u''' + u' = 0, x \in [0, L], \\ u(0) = u(L) = u'(0) = u'(L) = 0, \end{cases} \tag{17}$$

if and only if $L \in \mathcal{L}$, *where*

$$\mathcal{L} = \{L > 0 : G_{kl}(\theta^2) = 0,\ \theta = \frac{2\pi}{L},\ k > l > 0,\ k, l \in \mathbb{N}^+\}, \tag{18}$$

and

$$G_{kl}(y) = a^4k^2(k-l)^2l^2y^3+4a^2(k^2-kl+l^2)^2y^2-(a^2-3)(a^2+9)(k^2-kl+l^2)y-(9+a^2)^2. \tag{19}$$

Part of the proof of Theorem 3.1 is similar to the proof of Lemma 3.5 in [19].

Proof. Assume $0 \neq u \in H^3(0, L)$, which satisfies (17).

Then set $(\alpha, \beta) = (u''(0), u''(L)) \neq (0, 0)$. Let $v \in H^2(\mathbb{R})$ be the prolongation of u by 0.

We have

$$\lambda v - \lambda a^2 v'' + v''' + v' = \alpha\delta_0 - \beta\delta_L,\ \text{in } \mathscr{D}'(\mathbb{R}), \tag{20}$$

where δ_0 and δ_L are the Dirac measures at 0 and L, respectively.

It is obvious that there exists $\lambda \in \mathbb{C}$, and $0 \neq u \in H^3(0, L)$ satisfying (17) if and only if there exists complex α, β, λ with $(\alpha, \beta) \neq (0, 0)$ and a function $v \in H^2(\mathbb{R})$ with compact support in $[-L, L]$ such that equation (20) holds.

Taking Fourier transform of equation (20), we obtain

$$[\lambda(1 + a^2\xi^2) - i\xi^3 + i\xi]\hat{v}(\xi) = \alpha - \beta e^{-iL\xi}.$$

Clearly, $v \in \mathscr{D}'(\mathbb{R})$ has a compact support. By the Paley-Wiener theorem, $\hat{v}(\xi)$ is an entire function in $\mathbb{C}$. Explicitly,

$$\hat{v}(\xi) = -i\frac{\beta e^{-iL\xi} - \alpha}{\lambda(1 + a^2\xi^2) - i\xi^3 + i\xi}.$$

Since both $\beta e^{-iL\xi} - a$ and $\lambda(1 + a^2\xi^2) - i\xi^3 + i\xi$ are entire, $\hat{v}$ is entire if and only if roots of $\lambda(1 + a^2\xi^2) - i\xi^3 + i\xi$ are included in the roots of $\beta e^{-iL\xi} - \alpha$ counting multiplicity.

For $p > 1$, we say (see [2]) that the distribution $\mathbf{v} \in \mathscr{D}'(\mathbb{R})$ belongs to the space $H^0_p(\mathbb{R})$ if there is a function $v \in L^p(\mathbb{R})$ such that

$$\mathbf{v}(f) = \int_{\mathbb{R}} f(x) \cdot v(x)dx, \forall f \in \mathscr{D}(\mathbb{R}).$$

The Bessel potential space $H^s_p(\mathbb{R})$ is given by

$$H^s_p(\mathbb{R}) = \{v \in S'(\mathbb{R}); \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2}\hat{v}(\xi)] \in H^0_p(\mathbb{R})\},$$

where $\hat{v}$ is the Fourier transform of v and $\mathcal{F}^{-1}$ is the Fourier inverse transform.

The characterization of space of multipliers for Bessel Potential Spaces [2]

$$H^2_2(\mathbb{R}) = \{v \in S'(\mathbb{R}); \mathcal{F}^{-1}[(1 + |\xi|^2)\hat{v}(\xi)] \in H^0_2(\mathbb{R})\}.$$

Using the Paley-Wiener theorem (see VI.4. [25]) and the characterization of space of multipliers for Bessel Potential Spaces [2] $H^2_2(\mathbb{R})$, it suffices to show that there exists $\lambda \in \mathbb{C}$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus (0, 0)$ such that the map

$$f(\xi) = -i\frac{\beta e^{-iL\xi} - \alpha}{\lambda(1 + a^2\xi^2) - i\xi^3 + i\xi}$$

satisfies:

a. f is an entire function in $\mathbb{C}$;
b. $\int_{\mathbb{R}} |f(\xi)|^2(1 + |\xi|^2)^2 d\xi < \infty$;

c. $\xi \in \mathbb{C}$, $|f(\xi)| \leq C(1+|\xi|)^N e^{L|Im\xi|}$ for some positive constants C, N.

Notice that roots of $\beta e^{-iL\xi} - \alpha$ are simple and with periodic $\frac{2\pi}{L}$. Hence, if we set $\theta = \frac{2\pi}{L}$, there are complex number ξ_1 and positive integer $k > l > 0$ such that

$$\xi_2 = \xi_1 + k\theta \text{ and } \xi_3 = \xi_1 + l\theta,$$

and

$$\xi^3 + ia^2\lambda\xi^2 - \xi + i\lambda = (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3).$$

Conditions a. b. c. all hold if and only if there exists positive integer $k > l > 0$ and complex ξ_1 such that the following equation system has solution over $(\xi_1, \lambda) \in \mathbb{C}^2$ and $\theta > 0$:

$$\begin{cases} e_1 := (k+l)\theta + 3\xi_1 + ia^2\lambda = 0, \\ e_2 := kl\theta^2 + 2k\theta\xi_1 + 2l\theta\xi_1 + 3\xi_1^2 + 1 = 0, \\ e_3 := kl\theta^2\xi_1 + k\theta\xi_1^2 + l\theta\xi_1^2 + \xi_1^3 + i\lambda = 0. \end{cases} \tag{21}$$

To solve equation system (21) for given positive integer pairs $k > l > 0$, we consider the following invertible operations:

1. If we view e_2 and e_3 as polynomials in ξ_1, by Euclidian algorithm, we could eliminate ξ_1^3 and ξ_1^2 term in e_3, and obtain a remainder term r, which is linear in ξ_1. Thus, the re-scaled r is given by
$$r = (3 + 2(k^2 - kl + l^2)\theta^2)\xi_1 + (k+l)\theta(1 + kl\theta^2) - 9\lambda i = 0. \tag{22}$$
2. Solve λ from $e_1 = 0$, we obtain
$$\lambda = \frac{i((k+l)\theta + 3\xi_1)}{a^2}. \tag{23}$$
3. Plugging-in the previously solved λ into $r = 0$, we obtain a linear equation in ξ_1.
4. Solve the non-degenerate linear equation obtained from previous step for ξ_1, we obtain
$$\xi_1 = -\frac{(k+l)\theta(9 + a^2(1 + kl\theta^2))}{27 + a^2(3 + 2(k^2 - kl + l^2)\theta^2)}. \tag{24}$$
5. Plug in the previously solved ξ_1 in $e_2 = 0$.

After all steps mentioned above, we obtain

$$a^4k^2(k-l)^2l^2\theta^6 + 4a^2(k^2-kl+l^2)^2\theta^4 - (a^2-3)(a^2+9)(k^2-kl+l^2)\theta^2 - (9+a^2)^2 = 0. \tag{25}$$

Furthermore, the equation system (21) and equations (23), (24), and (25) have the same solutions, since steps 1-5 are all invertible. Later, we will show that equation (25) have unique positive solution for θ, thus ξ_1 is determined by (24), as well as from previous step 2 and step 4, we can also solve for

$$\lambda = \frac{i(k-2l)(2k-l)(k+l)\theta^3}{27 + a^2(3 + 2(k^2 - kl + l^2)\theta^2)} \subset i\mathbb{R}. \tag{26}$$

□

The explicit expression of the critical length L relies on the root formula for cubic equation (25) of θ^2, which is cumbersome. Instead, we will give a detailed analysis to narrow down its location.

Theorem 3.2. *For any integer pairs $k > l > 0$, $G_{kl}(y)$ has unique positive root on $(0, 1]$, denoted by y_{kl}. Thus, $L \in \mathcal{L}$ if and only if there exists integer pairs $k > l > 0$ such that $L = \frac{2\pi}{\sqrt{y_{kl}}}$.*

The root y_{kl} has the following properties for any $k > l > 0$:

1. *$\mathcal{L} \cap (0, 2\pi) = \emptyset$, and*
$$y_{kl} \in \left[\frac{3}{k^2 - kl + l^2}, \frac{c}{k^2 - kl + l^2}\right), \tag{27}$$
where $c = \frac{(a^2+9)(-3+a^2+(1+a^2)^{1/2}(9+a^2)^{1/2})}{8a^2} > 3$.
2. *The left bound is achieved when $\frac{l}{k} = \frac{1}{2}$ and right bound is asymptotically achieved, if $\frac{l(k-l)}{k^2} \to 0$. For any positive integer l, $2\pi l \in \mathcal{L}$.*
3. *The value of $y_{kl} \cdot (k^2 - kl + l^2)$ only depend on $\frac{l}{k}$ and a.*
4. *$\mathcal{L}$ is a nowhere dense set, and for any compact set K, $K \cap \mathcal{L}$ is a finite set.*

Proof. For any integers $k > l > 0$, it is easy to verify that

- $G_{kl}(0) < 0$,
- $G_{kl}(1) = a^4(k^2 - kl + 1)(1 + kl)(kl - l^2 - 1) + 27(k^2 - kl + l^2 - 3) + 2a^2(k^2 - kl + l^2 - 3)(2k^2 - 2kl + 2l^2 + 3) \geq 0$,
- $G_{kl}(1) = 0$ if and only if $k = 2, l = 1$,
- $G''_{kl}(y) = 6a^4k^2(k - l)^2l^2y + 8a^2(k^2 - kl + l^2)^2 > 0, \forall y \geq 0$, thus $G_{kl}(y)$ is strictly convex.

Therefore, we see that $G_{kl}(y)$ have unique positive root $y_{kl} \in (0, 1]$. Next, We will show properties 1, 2 and 3. $L \in \mathcal{L}$ if and only if there exists positive integers $k > l > 0$ such that $L = \frac{2\pi}{\sqrt{y_{kl}}} \geq 2\pi$, since $y_{kl} \leq 1$. Thus, $\mathcal{L} \cap (0, 2\pi) = \emptyset$.

Let
$$f(x, \gamma(\tau)) := G_{kl}\left(\frac{x}{k^2 - kl + l^2}\right) = -(9+a^2)^2 - (a^2-3)(a^2+9)x + 4a^2x^2 + a^4\gamma(\tau)x^3,$$
where $\tau = \frac{l}{k} \in (0, 1)$, and $\gamma(\tau) = \frac{(1-\tau)^2\tau^2}{(1-\tau+\tau^2)^3}$.

For $\tau \in (0, 1)$, we have $f(0, \gamma(\tau)) < 0, f(+\infty, \gamma(\tau)) = +\infty$, and $\frac{\partial^2}{\partial x^2}f(x, \gamma(\tau)) > 0$, which imply that $f(x, \gamma(\tau)) = 0$ have unique positive zero x_τ, which is continuous in τ. Then, $y_{kl} \cdot (k^2 - kl + l^2) = x_\tau$. This verifies property 3.

We can check directly that $f(3, \gamma(1/2)) = 0$, i.e., $x_{1/2} = 3$. This further implies that, when $k = 2l$,
$$L = \frac{2\pi}{\sqrt{y_{kl}}} = 2\pi\sqrt{\frac{3l^2}{3}} = 2l\pi \in \mathcal{L}.$$
On the other hand, $f(c, 0) = f(c, \gamma(0^+)) = f(c, \gamma(1^-)) = 0$ if
$$c = \frac{(a^2 + 9)(-3 + a^2 + (1 + a^2)^{1/2}(9 + a^2)^{1/2})}{8a^2}. \tag{28}$$
Since $\frac{\partial}{\partial\gamma}f(x, \gamma) > 0$ when $x > 0$, then $0 = f(c, 0) < f(c, \gamma(1/2))$. This, combined with $f(0, \gamma(1/2)) < 0$, implies that $0 < x_{1/2} < c$, i.e. $c > x_{1/2} = 3$.

To show (27), We only need to prove that $x_\tau \in [3, c)$ for $\tau \in (0, 1)$. Note that $\gamma(\tau)$ is symmetric about $\tau = 1/2$ and increasing on $(0, 1/2)$. Hence, its maximum value is $\gamma(1/2)$. Since $f(x, \gamma)$ is a strictly increasing function of γ for each x, we have $0 = f(x_{\tau_1}, \gamma(\tau_1)) > f(x_{\tau_1}, \gamma(\tau_2))$ when $\gamma(\tau_1) > \gamma(\tau_2)$. On the other hand, $f(c, \gamma(\tau_2)) > f(c, 0) = 0$. By the intermediate value theorem, there is a $x_{\tau_2} \in (x_{\tau_1},\ c)$ such that $f(x_{\tau_2}, \gamma(\tau_2)) = 0$. We conclude that x_τ is a decreasing function of $\gamma(\tau)$ for $\tau \in (0,1/2)$. Combining this with $x_0 = x_1 = c$, $x_{1/2} = 3$, we obtain the desired result.

Lastly, we will show property 4. Notice that $L \in \mathcal{L}$ if and only if there exists integers $k > l > 0$ such that $L = \frac{2\pi}{\sqrt{y_{kl}}}$. Thus,

$$\mathcal{L} \subset \bigcup_{k>l>0} \left(\frac{2\pi\sqrt{k^2 - kl + l^2}}{\sqrt{c}}, \frac{2\pi\sqrt{k^2 - kl + l^2}}{\sqrt{3}} \right].$$

Let K be any compact set in R and $M = \sup_{x\in K} |x|$. Assume that $k^2 \geq \frac{cM^2}{3\pi^2}$. Then, we have

$$\frac{2\pi\sqrt{k^2 - kl + l^2}}{\sqrt{c}} = \frac{2\pi\sqrt{(l-k/2)^2 + (3/4)k^2}}{\sqrt{c}} \geq \frac{2\pi\sqrt{(3/4)k^2}}{\sqrt{c}} \geq M, \tag{29}$$

which implies

$$K \cap \left(\frac{2\pi\sqrt{k^2 - kl + l^2}}{\sqrt{c}}, \frac{2\pi\sqrt{k^2 - kl + l^2}}{\sqrt{3}} \right] = \emptyset.$$

There are only finite number of positive integer pairs (k,l) such that $k > l > 0$ with $k^2 < \frac{cM^2}{3\pi^2}$, as well as for each pair (k,l) equation (19) only have one solution.

We conclude that that $K \cap \mathcal{L}$ is a finite set, hence $\mathcal{L}$ is a nowhere dense set. □

Remark: When $a = 0$, the linear KdV-BBM euqation become the linear KdV equation. Then (19) reduces to a linear equation, whose root has explicit expression $y_{kl} = \frac{3}{k^2-kl+l^2}$, $k > l > 0$. Thus,

$$\mathcal{L} = \left\{ 2\pi\sqrt{\frac{k^2 - kl + l^2}{3}}; k > l > 0, k,l \in \mathbb{N}^+ \right\} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}; k,l \in \mathbb{N}^+ \right\}, \tag{30}$$

which recovers the set of critical length for the linear KdV equation given in [15, 19].

4. **Exponential stability.** In order to prove the exponential stability, we recall the following known result (see [10, 11, 18]).

Theorem 4.1. *Let $\left\{e^{At}\right\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H. Then the semigroup is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ (resolvent set) and*

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to\infty} \left\|(i\lambda I - A)^{-1}\right\|_{\mathcal{L}(H)} < \infty. \tag{31}$$

The following Lemma is a key result which will be used to prove the exponential stability of the linear KdV-BBM equation. It gives the asymptotic behavior of the roots of the characteristic polynomial for the following ODE:

$$y'''(x) - ia^2 sy''(x) + y'(x) + isy(x) = 0.$$

Lemma 4.2. *Let $q_s(r) = r^3 - ia^2sr^2 + r + is$ be a polynomial, with root $r_j = r_j(s)$, $j = 1,2,3$. Then it has one imaginary root, denoted by r_3. Define $\mu_1 = \frac{1}{a}, \mu_2 = -\frac{1}{a}, \mu_3 = ia^2s$. Then, for $j = 1,2,3$, we have $r_j/\mu_j \to 1$, as $|s| \to \infty$.*

Proof. Since $iq_s(ir) = r^3 - a^2sr^2 - r - s$ is a third order polynomial with real coefficient, it has at least one real root, i.e., $q_s(r)$ has one imaginary root. Denote it by r_3.

Let $p_s(r) = (r-\mu_1')(r-\mu_2')(r-\mu_3')$ be another polynomial, with $\mu_1' = \frac{1}{a} - \frac{b_1}{s}i, \mu_2' = -\frac{1}{a} - \frac{b_1}{s}i$, and $\mu_3' = ia^2(s + \frac{b_2}{s})$. Let $\varepsilon = \frac{K}{|s|}$ and $C_j := \{z \in \mathbb{C} : |z - \mu_j'| = \varepsilon\}, j = 1, 2$. Now we are going to find a lower bound of $p_s(r)$ on each circle $C_j, j = 1, 2$. Let $\mu_j' + h \in C_j$.

Notice that with the choice of $b_1 = \frac{1}{2a^2} + \frac{1}{2a^4}$ and $b_2 = \frac{1}{a^4} + \frac{1}{a^6}$, we have the following estimation for large enough $|s|$:

$$\begin{aligned} &|p_s(\mu_1' + h)| = 2aK + O(\frac{1}{|s|}) \geq aK, \\ &|p_s(\mu_2' + h)| = 2aK + O(\frac{1}{|s|}) \geq aK, \\ &|p_s(z) - q_s(z)| = O(\frac{1}{|s|}) + O(\frac{1}{|s|^2})|z|. \end{aligned} \tag{32}$$

For any $z_j \in C_j$, $|z_j| = O(1)$. Thus, for large enough $|s|$,

$$|q_s(z_j) - p_s(z_j)| = O(\frac{1}{|s|}) \leq aK/2 < aK \leq |p_s(z_j)|, j = 1, 2.$$

Therefore, it follows the Rouche's Theorem that $(q_s(r) - p_s(r)) + p_s(r) = q_s(r)$ and $p_s(r)$ have the same number of zeros, counting multiplicities in the interior of C_j, i.e. $|r_j - \mu_j'| \leq \varepsilon, j = 1, 2$, which leads to $r_j - \mu_j' \to 0$, as $|s| \to \infty$, $j = 1, 2$.

Combined with $\mu_1' \to \mu_1, \mu_2' \to \mu_2$, we know that $r_j = r_j(s) \to \mu_j, j = 1, 2$, as $|s| \to \infty$.

As a consequence of the Vieta's formulas, we have

$$\frac{r_3}{\mu_3} = \frac{-is/(r_1 r_2)}{isa^2} \to 1,$$

as $|s| \to \infty$. □

Now, we present our main theorem for linear KdV-BBM equation.

Theorem 4.3. *If $L \notin \mathcal{L}$, then the semigroup e^{At} is exponentially stable.*

Proof. By Theorem 4.1, we need to verify the following two conditions

$$i\mathbb{R} \subset \rho(A) \tag{33}$$

and

$$\limsup_{s \in \mathbb{R}} \|(is - A)^{-1}\| < \infty. \tag{34}$$

We already know that $\sigma(A)$ only consists of eigenvalues of A, and $0 \notin \sigma(A)$. Suppose that is $(s \neq 0)$ is an imaginary eigenvalue of A, and $y \in D(T)$ is the corresponding eigenfunction, i.e.,

$$Ay = isy. \tag{35}$$

Then,

$$0 = \operatorname{Re} \langle isy, y \rangle_{\mathcal{H}} = \operatorname{Re} \langle Ay, y \rangle_{\mathcal{H}} = -\frac{1}{2}|y'(0)|^2, \tag{36}$$

which implies $y'(0) = 0$. This leads to

$$\begin{cases} y''' - ia^2 sy'' + y' + isy = 0, x \in (0, L), \\ y(0) = y(L) = y'(0) = y'(L) = 0. \end{cases} \tag{37}$$

As a consequence of Theorem 3.1, we know that y has to be zero since $L \notin \mathcal{L}$. Therefore, condition (33) holds.

Next, let's verify condition (34). This amounts to show that the solution of

$$(is - A)y = f \tag{38}$$

satisfies $\|y\|_{\mathcal{H}} \leq M\|f\|_{\mathcal{H}}$ for a constant $M > 0$, as $s \to \infty$. We will first assume that $f \in H_0^2(0, L)$, then apply the density argument for $f \in \mathcal{H}$.

Equation (38) can be written as the following boundary value problem of third order ODE

$$\begin{cases} y'''(x) - isy''(x) + y'(x) + isy(x) = f(x) - f''(x), \\ y(0) = y(L) = y'(L) = 0, y \in H^3(0, L), \end{cases} \tag{39}$$

whose solution is

$$y(x) = y_h(x) + y_p(x)$$

where $y_p(x)$ is a particular solution and $y_h(x)$ is the general solution to the associated homogeneous equation. Let $q_s(r) = r^3 - isa^2r^2 + r + is$ be the characteristics polynomial of (39). By Lemma 4.2, $q_s(r)$ has simple roots, namely r_1, r_2 and r_3 for s large enough. Thus,

$$y_h(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x), \tag{40}$$

where $y_i(x) = e^{r_i(x-L)}$, $i = 1, 2, 3$. It is easy to find the Wronskian

$$\begin{aligned} W[y_1, y_2, y_3](x) &= (r_3 - r_2)(r_2 - r_1)(r_3 - r_1)e^{(r_1+r_2+r_3)(x-L)} \\ &= (r_3 - r_2)(r_2 - r_1)(r_3 - r_1)e^{isa^2(x-L)}. \end{aligned}$$

For the simplicity of notation, we use cyclic notation for subscript, for example $r_{i+3} = r_i$. Denote $W_i(x) = y_{i+1}(x)y'_{i+2}(x) - y'_{i+1}(x)y_{i+2}(x)$. Then, by the variation of parameters formula, we have

$$\begin{aligned} y_p(x) &= \sum_{i=1}^{3} y_i(x) \int_L^x \frac{W_j(z)(f(z) - f''(z))}{W(z)} dz \\ &= \sum_{i=1}^{3} y_i(x) \int_L^x \frac{W_i(z)f(z)}{W(z)} dz + \sum_{i=1}^{3} y_i(x) \int_L^x \left(\frac{W_i(z)}{W(z)}\right)' f'(z)dz. \end{aligned} \tag{41}$$

Here, we have used the fact that, after integrating by parts, the boundary term at $z = x$ vanishes since the coefficient of $f'(x)$ is $\sum_{i=1}^{3} y_i(x)W_i(x) = 0$.

Furthermore,

$$\begin{aligned} y'_p(x) &= \sum_{i=1}^{3} y'_i(x) \int_L^x \frac{W_i(z)f(z)}{W(z)} dz + \sum_{i=1}^{3} y'_i(x) \int_L^x \left(\frac{W_i(z)}{W(z)}\right)' f'(z)dz \\ &= \sum_{i=1}^{3} r_i e^{r_i(x-L)} \int_L^x \frac{e^{-r_i(z-L)}[f(z) - r_i f'(z)]}{(-r_i + r_{i+1})(r_i - r_{j+2})} dz \end{aligned} \tag{42}$$

since $\displaystyle\sum_{i=1}^{3} y_i(x)W_j(x) = 0$, and

$$\sum_{i=1}^{3} y_i(x) \left(\frac{W_i(x)}{W(x)}\right)' = \left(\sum_{i=1}^{3} y_i(x) \frac{W_i(x)}{W(x)}\right)' - \frac{\sum_{i=1}^{3} y'_i(x)W_i(x)}{W(x)} = \left(\frac{W(x)}{W(x)}\right)' - \frac{0}{W(x)} = 0.$$

From (41-42), we have $y_p(L) = y_p'(L) = 0$. Therefore, applying the boundary conditions in (39) to $y(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + y_p(x)$ yields

$$\begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ e^{-r_1L} & e^{-r_2L} & e^{-r_3L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -y_p(0) \\ 0 \\ 0 \end{bmatrix}.$$

Let D_s be the determinant of above matrix. Then,

$$D_s = (e^{-Lr_2} - e^{-Lr_3})r_1 + (e^{-Lr_3} - e^{-Lr_1})r_2 + (e^{-Lr_1} - e^{-Lr_2})r_3.$$

Recall that

$$r_1 \sim \frac{1}{a}, \quad r_2 \sim -\frac{1}{a}, \quad r_3 \sim isa^2, \text{ and } \operatorname{Re}(r_3) = 0. \tag{43}$$

Hence, we obtain the estimates $|D_s| = O(|s|)$, and

$$\begin{cases} c_1 = \dfrac{-(r_2e^{-r_3L} - r_3e^{-r_2L})y_p(0)}{D_s} = O(y_p(0)), \\ c_2 = \dfrac{-(r_3e^{-r_1L} - r_1e^{-r_3L})y_p(0)}{D_s} = O(y_p(0)), \\ c_3 = \dfrac{-(r_1e^{-r_2L} - r_2e^{-r_1L})y_p(0)}{D_s} = O(\dfrac{y_p(0)}{s}), \\ c_1r_1, c_2r_2, c_3r_3 = O(y_p(0)). \end{cases} \tag{44}$$

It follows from (42)-(43), and the Hölder's inequality that

$$\|y_p\|_{\mathcal{H}} \leq M\|f\|_{\mathcal{H}}, \tag{45}$$

which further leads to, by Poincaré inequality, that

$$|y_p(0)| \leq M\|f\|_{\mathcal{H}}. \tag{46}$$

Due to the last line of (44), (46) and

$$y_h'(x) = c_1r_1y_1(x) + c_2r_2y_2(x) + c_3r_3y_3(x),$$

we obtain

$$\|y_h\|_{\mathcal{H}} \leq M\|f\|_{\mathcal{H}}. \tag{47}$$

Therefore, we have arrived at

$$\|y\|_{\mathcal{H}} \leq 2M\|f\|_{\mathcal{H}}. \tag{48}$$

This verifies condition (34). □

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